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Incomplete Information System and Its Optimal Selections

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Abstract—This paper deals with knowledge discovering in incomplete information systems (IIS). By an IIS, we mean a system with unknown data or partly-known data. This kind of system can be regarded as a set-valued system. The selections of an IIS are considered. The relationships between the reducts in the source system and in its selections are investigated. We also present the concept of maximum distribution reducts and optimal selections, from which we provide an approach to acquire decision rules from incomplete decision tables (IDT).

Keywords—Consistent set, Incomplete system, Reduction, Rough set.

1. INTRODUCTION

In the real world, many information systems (IS) are incomplete. By an incomplete information system (IIS), we mean a system with unknown data or partly-known data. In this situation, some attributes values may be subsets of attributes domain. This kind of system can be regarded as a set-valued system. Therefore, we can study the IIS via its selections.

Several solutions to the problem of knowledge discovering from an IIS have been proposed in the area of artificial intelligence [1–5]. The simplest among them consist in removing examples with nonsingle values or replacing them with the most common values. Through these methods, however, some valuable information may be lost in the process of knowledge acquisition. More complex approaches were presented in [4,5]. A Bayesian formalism is used in [4] to determine

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the probability distribution of unknown value over the possible values from the domain. It is suggested in [5] to predict an attribute value based on the values of other attributes of the object.

A rough set [6] approach to knowledge acquisition is based on the assumption that two objects are indiscernible with regard to attributes set if they have the same value for each attribute from the attributes set. In the case of IIS, such a definition is not sufficient.

Many types of rules' generation from IIS were investigated in the context of rough set. The approach from [4] consists in transforming an IIS to a complete system, where each object with an incomplete descriptor in the source system is represented by a set of possible subobjects in the target system. By using discernibility matrix and Boolean reasoning techniques [7,8], Kryszkiewicz [9–11] proposed an approach to compute reduction of knowledge and obtain optimal true and certain decision rules in incomplete decision tables (IDT). But the problem of partly-known values is not taken into consideration.

In this paper, we deal with the problem of knowledge discovering from IIS with unknown or partly-known values. The selections of the system are considered. The relationships between the reducts in the source system and in its selections are investigated. We can acquire decision rules from incomplete systems by their optimal selections.

2. IIS AND SET APPROXIMATIONS

An IIS is a triplet $\mathcal{I} = (U, A, F)$, where U is a nonempty, finite set of objects called the universe and A is a nonempty, finite set of attributes, such that $F(a, \cdot) : U \to \mathcal{P}_0(V_a)$ for any $a \in A$, where V_a is the finite domain of the attribute a, $\mathcal{P}(V_a)$ is the power set of V_a, and $\mathcal{P}_0(V_a)$ is $\mathcal{P}(V_a)$ except empty set $\emptyset$. If $F(a, \cdot) : U \to V_a$ is a single-valued mapping, then the system (U, A, F) is called complete IS. It is easy to understand that a complete IS can also be regarded as an IIS.

Let $\mathcal{S} = (U, A, f)$ be a complete IS. Each nonempty subset $B \subseteq A$ determines an indiscernibility relation [6]

$$R_B^f = \{(x, y) \in U \times U : f(a, x) = f(a, y),\ \forall\, a \in B\}. \tag{1}$$

R_B^f partitions U into equivalence classes

$$U/R_B^f = \left\{[x]_B^f : x \in U\right\}, \tag{2}$$

where $[x]_B^f$ denotes the equivalence class determined by x with respect to B, i.e.,

$$[x]_B^f = \left\{y \in U : (x, y) \in R_B^f\right\}. \tag{3}$$

Objects from $[x]_B^f$ are indiscernible with regard to their descriptions in the system.

Let $\mathcal{I} = (U, A, F)$ be an IIS. Each nonempty subset $B \subseteq A$ determines a similarity relation

$$\mathrm{SIM}(B) = \{(x, y) \in U \times U : F(a, x) \cap F(a, y) \neq \emptyset,\ \forall\, a \in B\}. \tag{4}$$

Similarity relation $\mathrm{SIM}(B)$ is reflexive and symmetric, but not transitive. By $S_B(x)$, we denote the set of objects $\{y \in U :\ (x, y) \in \mathrm{SIM}(B)\}$. $S_B(x)$ can be treated as a neighborhood of x [12,13]. Objects from $S_B(x)$ may have the same description with x in the system.

For any complete IS $(U, A, f), B \subseteq A$, $x \in U$, it is easy to see that $\mathrm{SIM}(B) = R_B^f$, and $S_B(x) = [x]_B^f$.

Letting $X \subseteq U$, $B \subseteq A$, one can characterize X by a pair of upper and lower approximations:

$$\begin{aligned} \overline{\mathrm{SIM}(B)}(X) &= \{x \in U : S_B(x) \cap X \neq \emptyset\}, \\ \underline{\mathrm{SIM}(B)}(X) &= \{x \in U : S_B(x) \subseteq X\}. \end{aligned} \tag{5}$$

The lower approximation $\underline{\mathrm{SIM}(B)}(X)$ is the set of objects that belong to X with certainty regardless of the actual values of unknown and partly-known attributes, whereas the upper approximation $\overline{\mathrm{SIM}(B)}(X)$ is the set of objects that possibly belong to X for some actual values of unknown or partly-known attributes. Similar to [11], we have the following results.

THEOREM 2.1. *Let $\mathcal{I} = (U, A, F)$ be an IS, and $B \subseteq A$. Then $\underline{\mathit{SIM}(B)}$ and $\overline{\mathit{SIM}(B)}$ satisfy the following properties: $\forall X, Y \in \mathcal{P}(U)$,*

(1) $\underline{\mathit{SIM}(B)}(X) \subseteq X \subseteq \overline{\mathit{SIM}(B)}(X)$;

(2) $\underline{\mathit{SIM}(B)}(\sim X) = \sim \overline{\mathit{SIM}(B)}(X)$;

(3) $\underline{\mathit{SIM}(B)}(U) = U = \overline{\mathit{SIM}(B)}(U)$, $\underline{\mathit{SIM}(B)}(\emptyset) = \emptyset = \overline{\mathit{SIM}(B)}(\emptyset)$;

(4) $\overline{\mathit{SIM}(B)}(X \cup Y) = \overline{\mathit{SIM}(B)}(X) \cup \overline{\mathit{SIM}(B)}(Y)$, $\underline{\mathit{SIM}(B)}(X \cap Y) = \underline{\mathit{SIM}(B)}(X) \cap \underline{\mathit{SIM}(B)}(Y)$;

(5) $X \subseteq Y \Longrightarrow \underline{\mathit{SIM}(B)}(X) \subseteq \underline{\mathit{SIM}(B)}(Y)$, $\overline{\mathit{SIM}(B)}(X) \subseteq \overline{\mathit{SIM}(B)}(Y)$;

(6) $X \subseteq \underline{\mathit{SIM}(B)}\,\overline{\mathit{SIM}(B)}(X)$, $\overline{\mathit{SIM}(B)}\,\underline{\mathit{SIM}(B)}(X) \subseteq X$.

PROOF. It is trivial. ∎

DEFINITION 2.1. *Let $\mathcal{I} = (U, A, F)$, $\mathcal{S}^f = (U, A, f)$. We say that $\mathcal{S}^f$ is a selection of $\mathcal{I}$, if $\mathcal{S}^f$ is complete and $f(a, x) \in F(a, x)$, $\forall\, a \in A$, $x \in U$. Sometimes, a selection of $\mathcal{I}$ is also called a completion of $\mathcal{I}$. The set of all selections of $\mathcal{I}$ will be denoted by $S_{\mathcal{I}}$.*

THEOREM 2.2. *Let $\mathcal{I} = (U, A, F)$ be an IIS and $B \subseteq A$. Then,*

(1) $S_B(x) = \bigcup_{\mathcal{S}^f \in S_{\mathcal{I}}} [x]^f_B$, $\forall x \in U$;

(2) $\mathit{SIM}(B) = \bigcup_{\mathcal{S}^f \in S_{\mathcal{I}}} R^f_B$.

PROOF OF (1). It follows immediately from the definitions.

PROOF OF (2). $\forall \mathcal{S}^f \in S_{\mathcal{I}}$, we have that

$$\begin{aligned} R^f_B &= \{(x, y) \in U \times U : f(b, x) = f(b, y), \forall\, b \in B\} \\ &\subseteq \{(x, y) \in U \times U : F(b, x) \cap F(b, y), \forall\, b \in B\} = \mathrm{SIM}(B). \end{aligned}$$

Conversely, $\forall\,(x, y) \in \mathrm{SIM}(B)$, we have that $F(a, x) \cap F(a, y) \neq \emptyset$, $\forall\, a \in B$. Then we can select $z(a, x, y) \in F(a, x) \cap F(a, y)$. It is easy to see that there is an information function $f : A \times U \longrightarrow V$ such that $f(a, x) = f(a, y) = z(a, x, y)$. Therefore, $\mathcal{S}^f \in S_{\mathcal{I}}$ and $(x, y) \in R^f_B$. Hence, $\mathrm{SIM}(B) \subseteq \bigcup_{\mathcal{S}^f \in S_{\mathcal{I}}} R^f_B$.

Thus, we complete the proof of (2). ∎

3. COMPUTATION OF REDUCTS IN CONSISTENT IDT

Let (U, A, F) be an IS. Then, (U, A, F, d) is called a decision table (DT), where $d \notin A$ is a distinguished attribute called the decision attribute such that $d : U \longrightarrow V_d$, and the elements of A are called the condition attributes. If (U, A, F) is complete, then (U, A, F, d) is called a complete decision table, otherwise, it is incomplete.

Letting (U, A, f) be a selection of (U, A, F), we say that the complete DT (U, A, f, d) is a selection of an IDT, $\mathcal{I} = (U, A, F, d)$. The set of all selections of $\mathcal{I}$ is denoted by $S_{\mathcal{I}}$.

If $\mathrm{SIM}(A) \subseteq R_d$, then $\mathcal{I} = (U, A, F, d)$ is consistent, otherwise, it is inconsistent. For $B \subseteq A$, if $\mathrm{SIM}(B) \subseteq R_d$, we say that B is a consistent set of $\mathcal{I}$. If B is a consistent set of $\mathcal{I}$ and $\forall\, b \in B$, $B \setminus \{b\}$ is not a consistent set, then B is referred to as a reduct of $\mathcal{I}$.

Invoking Theorem 2.2, we have the following relationships between the reducts in the selections and in the source system.

THEOREM 3.1. *$\mathcal{I} = (U, A, F, d)$ is a consistent IDT iff every selection of $\mathcal{I}$ is consistent.*

THEOREM 3.2. *Let $\mathcal{I} = (U, A, F, d)$ be a consistent IDT, $B \subseteq A$. Then B is a consistent set of $\mathcal{I}$ iff B is a consistent set of every selection of $\mathcal{I}$.*

Table 1. A consistent IDT.

U	a_1	a_2	a_3	a_4	d
x_1	$\{1\}$	$\{1,2\}$	$\{2\}$	$\{1\}$	1
x_2	$\{2\}$	$\{1\}$	$\{1,2\}$	$\{1\}$	1
x_3	$\{2\}$	$\{2\}$	$\{2\}$	$\{1\}$	2
x_4	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	2

THEOREM 3.3. *Let $\mathcal{I} = (U, A, F, d)$ be a consistent IDT, $B \subseteq A$. If B is a reduct of every selection of $\mathcal{I}$, then B is a reduct of $\mathcal{I}$.*

The following example illustrates that the converse of Theorem 3.3 is not true.

EXAMPLE 3.1. A reduct of consistent decision table $\mathcal{I}$ may not be a reduct of the selection of $\mathcal{I}$ (Table 1).

It is easy to verify that $\{a_1, a_2, a_3\}$ is a reduct of $\mathcal{I}$, but not a reduct of any selection of $\mathcal{I}$.

Now we present an approach to compute reducts in consistent IDTs.

THEOREM 3.4. *Let (U, A, F, d) be a consistent IDT, and $B \subseteq A$. Denote*

$$D(x,y) = \begin{cases} \{a \in A : F(a,x) \cap F(a,y) = \emptyset\}, & d(x) \neq d(y), \\ \emptyset, & d(x) = d(y). \end{cases} \tag{6}$$

$$\mathbf{D} = \{D(x,y) : D(x,y) \neq \emptyset\}.$$

Then the following conditions are equivalent:

(1) *$SIM(B) \subseteq R_d$;*
(2) *$\forall D(x,y) \in \mathbf{D}, D(x,y) \cap B \neq \emptyset$.*

PROOF. "(1)⇒(2)" Suppose $\mathrm{SIM}(B) \subseteq R_d$.

If $D(x,y) \neq \emptyset$, then $d(x) \neq d(y)$, thus, $y \notin [x]_{\{d\}}$, where $[x]_{\{d\}}$ denotes the decision class determined by x. Noticing that $\mathrm{SIM}(B) \subseteq R_d$, we have $y \notin S_B(x)$. Therefore, there exists $b \in B$ such that $F(b,x) \cap F(b,y) = \emptyset$. That is, $b \in B \cap D(x,y)$.

"(2)⇒(1)" It needs only to prove that $\forall x \in U$, we have $S_B(x) \subseteq [x]_{\{d\}}$.

In fact, for all $y \in S_B(x)$, we have the following.

If $d(x) \neq d(y)$, then $[x]_{\{d\}} \cap [y]_{\{d\}} = \emptyset$. Because $S_A(x) \subseteq [x]_{\{d\}}$ and $S_A(y) \subseteq [y]_{\{d\}}$, we have $S_A(x) \cap S_A(y) = \emptyset$. Thus, there exists $a \in A$ such that $F(a,x) \cap F(a,y) = \emptyset$. Therefore, $D(x,y) \neq \emptyset$. From Condition (2), we have $B \cap D(x,y) \neq \emptyset$. It follows that there exists $b \in B$ such that $F(b,x) \cap F(b,y) = \emptyset$. Thus, $y \notin [x]_B$, a contradiction.

Therefore, $d(x) = d(y)$, it must have $[y]_{\{d\}} = [x]_{\{d\}}$ and then $y \in [x]_{\{d\}}$. Thus, we complete the proof. ∎

Let $\sum D(x,y)$ be a disjunction of variables corresponding to attributes contained in $D(x,y)$, if $D(x,y) \neq \emptyset$. Otherwise, let $\sum D(x,y)$ be a Boolean expression which is equal to 1. Then we have the following result.

THEOREM 3.5. *Let $\mathcal{I} = (U, A, F, d)$ be a consistent IIS. Then an attribute subset B of A is a reduct of $\mathcal{I}$ iff $\wedge B$ is a prime implicant of discernibility function*

$$\Delta = \prod_{D(x,y) \in \mathbf{D}} \sum D(x,y). \tag{7}$$

PROOF. It follows directly from Theorem 3.4.

Table 2. A consistent IDT.

U	a_1	a_2	a_3	a_4	d
x_1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	0
x_2	$\{1\}$	$\{1,3\}$	$\{1\}$	$\{1\}$	0
x_3	$\{1,2\}$	$\{2\}$	$\{1,2\}$	$\{2\}$	1
x_4	$\{2\}$	$\{1\}$	$\{1\}$	$\{2\}$	1
x_5	$\{2\}$	$\{1,2\}$	$\{2\}$	$\{2\}$	1

EXAMPLE 3.2. Table 2 is a consistent IDT.

All the nonempty $D(x_i, x_j)$ are

$$\begin{array}{lll} D(x_1,x_3) = \{a_2,a_4\}, & D(x_1,x_4) = \{a_1,a_4\}. & D(x_1,x_5) = \{a_2,a_3,a_4\}, \\ D(x_2,x_3) = \{a_2,a_4\}, & D(x_2,x_4) = \{a_1,a_4\}, & D(x_2,x_5) = \{a_2,a_3,a_4\}. \end{array}$$

The discernibility function is

$$\begin{aligned} \triangle &= (a_2 \vee a_4) \wedge (a_1 \vee a_4) \wedge (a_1 \vee a_3 \vee a_4) \\ &= (a_1 \wedge a_2) \vee a_4. \end{aligned}$$

Therefore, $\{a_1, a_2\}$ and $\{a_4\}$ are two reducts of the IDT.

4. THE OPTIMAL SELECTIONS OF INCONSISTENT IDT

For a consistent DT $\mathcal{I} = (U, A, F, d)$, we can acquire certainty decision rules from the description of each subject in the universe. But if the DT is not consistent, we may not derive a set of certain definite rules covering all objects with confidence 1. In this section, we present an approach to acquire decision rules from inconsistent IDT. First, we present a definition of maximum distribution reduct.

DEFINITION 4.1. *Let $\mathcal{I} = (U, A, F, d)$ be a DT, and $B \subseteq A$. Denoted by*

$$U/R_{\{d\}} = \{D_1, \dots, D_r\}.$$

$$m_B(x) = \max\{D(D_j/S_B(x)) : j \le r\} = D(D_{j_i}/S_B(x)), \qquad x \in U. \tag{8}$$

where for $E, F \in \mathcal{P}(U)$, $D(E/F) = |E \cap F|/|F|$ if the cardinality of F, $|F| \neq 0$, otherwise, $D(E/F) = 1$. It is easy to see that $D(\cdot/\cdot)$ is an inclusion degree on $\mathcal{P}(U)$ (see [14]).

The maximum decision function γ_B is defined as follows:

$$\gamma_B(x) = \{D_{j_i} : D(D_{j_i}/S_B(x)) = m_B(x)\}, \qquad x \in U. \tag{9}$$

If $\gamma_B(x) = \gamma_A(x)$ for all $x \in U$, we say that B is a maximum distribution consistent set of (U, A, F, d). If B is a maximum distribution consistent set, and no proper subset of B is maximum distribution consistent, then B is referred to as a maximum distribution reduct of (U, A, F, d).

A maximum distribution consistent set preserves all maximum decision classes. But the degree of confidence of each decision rule derived from the reduced system may not be equal to the one derived from the original system supported by the same object.

For a complete decision table (U, A, f, d), the concept of maximum distribution reduct is defined by Zhang, Mi and Wu (see [15]), which is a special case in an incomplete decision table. For the sake of simplicity, we denote by $m_B^f(x)$ and $\gamma_B^f(x)$ instead of $m_B(x)$ and $\gamma_B(x)$, respectively, in the complete DT (U, A, f, d). Now we compare the maximum distribution reduct in an IDT with that in its selections.

EXAMPLE 4.1. Continuing from Example 3.1, we can verify also that $\{a_1, a_2, a_3\}$ is a maximum distribution reduct of $\mathcal{I}$, but not a maximum distribution reduct of any selection of $\mathcal{I}$. While $\{a_1, a_3\}$ is a maximum distribution reduct of $\mathcal{S}$, where $\mathcal{S}$ is a selection of $\mathcal{I}$ in which $a_2(x_1) = 1$ and $a_3(x_2) = 1$. But $\{a_1, a_3\}$ is not a maximum distribution reduct of $\mathcal{I}$.

In addition, a maximum distribution consistent set of a selection of $\mathcal{I}$ may not be a maximum distribution consistent set of $\mathcal{I}$ (see also Example 3.1). Conversely, a maximum distribution consistent set of $\mathcal{I}$ may not be a maximum distribution consistent set of a selection of $\mathcal{I}$ (see Example 4.2).

EXAMPLE 4.2. Table 3 illustrates that a maximum distribution consistent set of $\mathcal{I}$ may not be a maximum distribution consistent set of a selection of $\mathcal{I}$.

Table 4 is a selection S^f of $\mathcal{I}$. We can verify that $\{a_1, a_2\}$ is a maximum distribution consistent set of $\mathcal{I}$, but not a maximum distribution consistent set of the selection S^f of $\mathcal{I}$.

Table 3. An inconsistent IDT $\mathcal{I}$.

U	a_1	a_2	a_3	d
x_1	$\{1\}$	$\{1,2\}$	$\{1\}$	1
x_2	$\{1\}$	$\{1\}$	$\{1,2\}$	2
x_3	$\{1\}$	$\{2\}$	$\{1\}$	2
x_4	$\{2\}$	$\{1\}$	$\{2\}$	1

Table 4. A selection of $\mathcal{I}$.

U	a_1	a_2	a_3	d
x_1	1	1	1	1
x_2	1	1	2	2
x_3	1	2	1	2
x_4	2	1	2	1

DEFINITION 4.2. *Let $\mathcal{I} = (U, A, F, d)$ be a decision table, $\mathcal{S}^f = (U, A, f, d)$ be a selection of $\mathcal{I}$, and B_f a maximum distribution reduct of $\mathcal{S}^f$ [15]. The reduced complete decision table (U, B_f, f, d) is called an optimal complete selection of $\mathcal{I}$, if B_f has the smallest cardinality in all maximum distribution reducts of all selections of $\mathcal{I}$ such that $\min_{x\in U} m^f_{B_f}(x) = \max \min_{x\in U} m^g_{B_g}(x)$, where the maximum is taken over all selections of $\mathcal{I}$ and all maximum distribution reducts of the selections.*

Now we present the process of rules acquisition in an IDT $\mathcal{I} = (U, A, F, d)$.

STEP 1. Find all selections of $\mathcal{I} = (U, A, F, d)$.

STEP 2. Find all maximum distribution reducts of all selections of the decision table $\mathcal{I}$.

The approach to find all maximum distribution reducts of a complete decision table can be found in [15].

STEP 3. For each maximum distribution reduct B_f of a selection (U, A, f, d), compute $m^f_{B_f}(x) = \max\{D(D_j/[x]^f_{B_f}) : j \le r\} = D(D_{j_i}/[x]^f_{B_f})$, $x \in U$.

STEP 4. Compute $m = \max \min_{x\in U} m^f_{B_f}(x)$, where the maximum is taken over all selections of (U, A, F, d) and all maximum distribution reducts of the selections.

STEP 5. Find a selection (U, A, f, d) and its reduct B_f such that $\min_{x\in U} m^f_{B_f}(x) = m$. Then the reduced complete decision table (U, B_f, f, d) is the optimal complete selection of (U, A, F, d), from which we can acquire decision rules of the original decision table.

Table 5. An inconsistent IDT.

U	a	b	c	d
x_1	$\{1\}$	$\{1\}$	$\{1\}$	1
x_2	$\{1\}$	$\{1,2\}$	$\{1\}$	1
x_3	$\{2\}$	$\{1\}$	$\{1\}$	1
x_4	$\{1\}$	$\{2\}$	$\{1,2\}$	1
x_5	$\{1\}$	$\{1,2\}$	$\{1\}$	2
x_6	$\{2\}$	$\{2\}$	$\{2\}$	2
x_7	$\{1\}$	$\{1\}$	$\{1\}$	2

Table 6. Selections of $\mathcal{I}$.

S^1				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	1	1	1
x_3	2	1	1	1
x_4	1	2	1	1
x_5	1	1	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^2				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	1	1	1
x_3	2	1	1	1
x_4	1	2	1	1
x_5	1	2	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^3				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	1	1	1
x_3	1	2	2	1
x_4	1	2	1	1
x_5	1	1	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^4				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	1	1	1
x_3	2	1	1	1
x_4	1	2	2	1
x_5	1	2	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^5				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	2	1	1
x_3	2	1	1	1
x_4	1	2	1	1
x_5	1	1	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^6				
U	a	b	c	d
x_1	1	1	1	
x_2	1	2	1	1
x_3	2	1	1	1
x_4	1	2	1	1
x_5	1	2	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^7				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	2	1	1
x_3	2	1	1	1
x_4	1	2	2	1
x_5	1	1	1	2
x_6	2	2	2	2
x_7	1	1	1	2

S^8				
U	a	b	c	d
x_1	1	1	1	1
x_2	1	2	1	1
x_3	2	1	1	1
x_4	1	2	2	1
x_5	1	2	1	2
x_6	2	2	2	2
x_7	1	1	1	2

EXAMPLE 4.3. (See [10].) Table 5 is an inconsistent IDT $\mathcal{I} = (U, A, F, d)$.

All the selections of $\mathcal{I}$ are presented in Table 6.

We can verify that $\{a, b\}$ is the unique maximum distribution reduct of S^1, S^5, S^6, and S^7. $\{a, b, c\}$ is a unique maximum distribution reduct of S^4 and S^8. The maximum distribution reduct of S^2 is $\{a, b\}$ and $\{b, c\}$, and the maximum distribution reduct of S^3 is $\{a, b\}$ and $\{a, c\}$.

After computing, we know that the reduced selections S^5 and S^7 are optimal, from which we can acquire the following decision rules:

$$(a,1) \wedge (b,1) \longrightarrow (d,2), \qquad (a,1) \wedge (b,2) \longrightarrow (d,1),$$
$$(a,2) \wedge (b,1) \longrightarrow (d,1), \qquad (a,2) \wedge (b,2) \longrightarrow (d,2).$$

5. CONCLUSIONS

Classical definitions of lower and upper approximations were originally introduced by Pawlak with reference to an indiscernibility relation which was assumed to be an equivalence relation. This model is useful in the analysis of data presented in terms of complete information systems. But if some of the attributes values are not known or partly known, there is no available equivalence relation from the system. A similarity relation is useful to express several interesting properties of this incomplete system. In particular, they allow the expression of the relationships between the source system and its completions. In this paper, we compared the reducts between the source system and its selections. By introducing the concept of maximum distribution reduct, we presented a process of finding decision rules from incomplete decision tables. Further research of knowledge acquisition for different requirements in incomplete systems is still needed.

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